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A NEW LOOK AT SHEAR CORRECTION FACTORS AND WARPING FUNCTIONS OF ANISOTROPIC LAMINATES

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Abstract—Shear warpings of a composite laminate are shown to be functions of its stacking sequence, its ply angles, and the material properties of layers. Moreover, a new formulation of shear correction factors is derived by matching the exact shear stress resultants and shear strain energy with those of the equivalent first-order shear deformation theory. Without solving a plate problem with specified loading and boundary conditions, approximate analytical shear warping functions of a composite laminate can be derived by using the continuity conditions of in-plane displacements and interlaminar shear stresses and the free shear stress conditions on the bonding surfaces. The approach results in a layerwise higher-order shear deformation theory that contains most shear deformation theories as special cases and reveals the shear coupling effect of angle-ply laminates. A combination of the present method of deriving shear warping functions and the new formulation of shear correction factors can be used to obtain fairly accurate *a priori* estimates of shear correction factors for the use of the first-order shear deformation theory in analyzing highly anisotropic laminates. For isotropic plates, the shear correction factor obtained is 5/6. For orthotropic laminates, the shear correction factors obtained are compared with those results in the literature that are obtained by using predictor–corrector procedures. Shear correction factors for some symmetric and asymmetric angle-ply laminates, of which the shear correction factor accounting for the shear coupling effect is non-trivial, are also presented.

1. INTRODUCTION

Shear effects are significant for composite laminates, since the ratios of the Young moduli to the shear moduli are between 20 and 50 in modern composites and between 2.5 and 3.0 in isotropic materials. Moreover, transverse shear deformation effects can be more important in anisotropic plates than in isotropic or even orthotropic structures of the same geometry (Gulati and Essenburg, 1967; Noor and Burton, 1990a). Although transverse shear stresses are much smaller than in-plane stresses, they cause delamination because laminated composites are weak in interlaminar shear strengths. Hence an accurate prediction of interlaminar shear stresses is necessary in the failure analysis of laminated composites.

For thin laminated plates, lamination theories that employ a single expansion for the warping displacements throughout the entire laminated thickness are commonly used in dynamic analysis. There are several shear-deformable plate theories, such as the first-order, third-order (Bhimaraddi, 1984; Reddy and Liu, 1985; Dennis and Palazotto, 1989), and other higher-order theories (Mirsky and Herrmann, 1957; Zukas and Vinson, 1971; Whitney and Sun, 1974; Voyiadjis and Shi, 1991). However, these single-expansion theories do not account for the continuity of interlaminar shear stresses and the elastic coupling of the two transverse shear deformations, and hence they are valid only for isotropic and one-layer orthotropic plates (Pai *et al.*, 1993; Pai and Nayfeh, 1994).

Although exact linear solutions of some laminated plates can be obtained by solving the three-dimensional elasticity equations, the procedure becomes very cumbersome when the number of layers increases (Pagano, 1969, 1970). Moreover, the resulting warping functions are functions of loading and boundary conditions and thickness-to-span ratio. Hence the use of three-dimensional models in analyzing anisotropic plates is computationally expensive and infeasible for practical use.

After the global responses of a laminate are obtained by using either the classical plate theory or the first-order shear deformation theory, indirect post-processing techniques using the three-dimensional elasticity equations can give reasonable solutions for interlaminar stresses (Whitney, 1987). For anisotropic plates, the use of the classical plate theory results in inaccurate global responses. On the other hand, because the first-order shear deformation theory requires shear correction factors to adjust the transverse shear stiffnesses to match the response predicted by the two-dimensional theory with that of the three-dimensional elasticity theory, accurate results can be obtained by using the first-order shear deformation theory only if correct shear correction factors are used. Unfortunately, shear correction factors are unknown before exact solutions are obtained. Hence there is concern in using these indirect post-processing techniques in general application, especially for highly anisotropic plates.

To improve the indirect post-processing techniques, Noor and Burton (1989a, 1990a,b) and Noor *et al.* (1990) proposed predictor–corrector procedures to correct the shear correction factors used in the first-order shear theory by using iteration processes. First, the first-order shear deformation theory is used in the predictor phase to calculate initial estimates for the gross structural response as well as the in-plane stresses. Second, three-dimensional equilibrium equations and constitutive relations are used to calculate transverse shear and normal stresses and strains. Third, the results from solving the three-dimensional equations are used to correct the shear correction factors or to obtain shear warping functions. Finally, the system is analyzed again by using the modified model. Although this method is likely to be able to obtain correct interlaminar stresses, solving three-dimensional elasticity equations and iteration procedures are computationally complex and expensive. Moreover, the shear correction factors obtained are functions of boundary conditions, plate geometry, and loading conditions, and hence they cannot be directly used for other plate configurations. Moreover, this method cannot be easily extended to solve problems involving geometric non-linearities.

Several approaches have been proposed for obtaining the shear correction factors for composite laminates. Most of these approaches are based on matching certain gross responses predicted by the first-order theory with those obtained from the three-dimensional elasticity theory. Gross responses used for matching include the transverse shear strain energy, the propagation velocity of a flexural wave, the natural frequency of the thickness shear vibration mode, etc. (Yang *et al.*, 1966; Chow, 1971; Dong and Tso, 1972; Whitney, 1973; Bert, 1983). All these methods require the elasticity equilibrium equations to be solved.

Since the results of Noor *et al.* (1990) show that only one iteration is usually required for obtaining accurate results even with a rough initial guess of the shear correction factors, the first-order shear deformation theory seems to be adequate for predicting the gross response characteristics of plates without iteration if fairly accurate shear correction factors are used in the first run. The first-order shear deformation theory has several major advantages over other theories: (i) only five displacement variables are used in the modeling; (ii) only linear functions of the thickness co-ordinate are involved in the displacement field, and hence exact structural matrices in a finite-element analysis can be obtained without using direct numerical integrations; and (iii) only C^0 continuity is required for the shear variables if the influence of shear deformations on the in-plane strains is neglected. Since these advantages are very useful in developing a large-scale finite-element code of analyzing anisotropic composites, it is also useful to obtain approximated shear warping functions and shear correction factors before solving the three-dimensional equilibrium equations. However, the problem is how to obtain shear warping functions and shear correction factors without solving the whole system with specified boundary and loading conditions.

For moderately thick composite shells, Di Sciuva (1987) used a piecewise linear displacement field to fulfill the interlaminar continuity conditions for shear stresses. Unfortunately, this theory does not satisfy the free shear-stress conditions on the bonding surfaces. Pai *et al.* (1993) and Pai and Nayfeh (1994) extended the piecewise linear displacement field used by Di Sciuva (1987) by using quadratic and cubic interpolation functions. This displacement field satisfies the continuity conditions of interlaminar shear stresses, accommodates free shear-stress conditions on the bonding surfaces, and accounts for initial curvatures and non-uniform distributions of transverse shear stresses in each layer. This theory contains most shear theories as special cases and reveals the shear coupling effect.

In this paper, we modify our layerwise higher-order shear-deformation theory (Pai *et al.*, 1993; Pai and Nayfeh, 1994) and investigate the characteristics of shear warping functions and the shear coupling effect. Moreover, we present a general derivation and interpretation of shear correction factors of anisotropic laminates. A combination of these two new developments can be used to obtain a better estimation of shear correction factors before solving for the response of an anisotropic laminate to specified loadings and boundary conditions.

2. SHEAR WARPING FUNCTIONS

To show that shear strains are independent of the reference plane deformations, we consider the Kirchhoff bending of a plate. Figure 1 shows the co-ordinate system xyz, the plate thickness h, and the ply angle α. With the adoption of Kirchhoff hypothesis, the displacement field is

$$\begin{aligned} u_1 &= u - w_{,x} z \\ u_2 &= v - w_{,y} z \\ u_3 &= w \end{aligned} \tag{1}$$

where u, v, and w are displacements of the reference point (i.e. the point on the reference plane) along the axes x, y, and z, respectively, and u_1, u_2, and u_3 are displacements of an arbitrary point on the observed plate element along the axes x, y, and z, respectively. Transverse shear strains ε_{13} and ε_{23} are obtained as

$$\varepsilon_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = 0, \quad \varepsilon_{23} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = 0. \tag{2}$$

Hence, under the assumption of Kirchhoff bending, there are no transverse shear strains. Moreover, the constitutive equation for the ith layer in a laminate is (Whitney, 1987)

$$\begin{Bmatrix} \sigma_{13}^{(i)} \\ \sigma_{23}^{(i)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^{(i)} & \bar{Q}_{45}^{(i)} \\ \bar{Q}_{45}^{(i)} & \bar{Q}_{44}^{(i)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{13}^{(i)} \\ \varepsilon_{23}^{(i)} \end{Bmatrix} \tag{3}$$

where

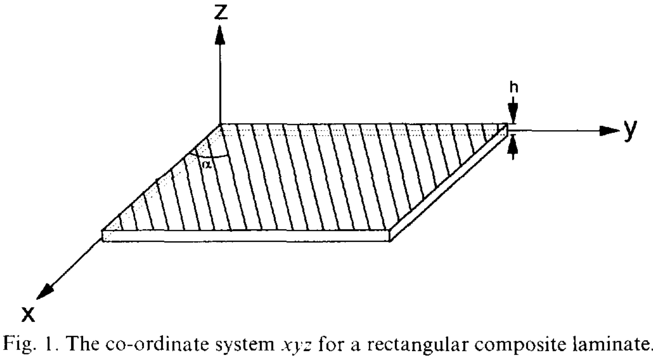


Fig. 1. The co-ordinate system xyz for a rectangular composite laminate.

$$\bar{Q}_{44}^{(i)} = \cos^2\alpha^{(i)} G_{23}^{(i)} + \sin^2\alpha^{(i)} G_{13}^{(i)}$$
$$\bar{Q}_{55}^{(i)} = \sin^2\alpha^{(i)} G_{23}^{(i)} + \cos^2\alpha^{(i)} G_{13}^{(i)}$$
$$\bar{Q}_{45}^{(i)} = \sin\alpha^{(i)} \cos\alpha^{(i)} (G_{13}^{(i)} - G_{23}^{(i)}) \tag{4}$$

and where $G_{13}^{(i)}$ and $G_{23}^{(i)}$ are transverse shear moduli and $\alpha^{(i)}$ is the ply angle of the ith layer. Hence it follows from eqns (2) and (3) that transverse shear stresses $\sigma_{13}^{(i)}$ and $\sigma_{23}^{(i)}$ are also zero.

Since reference plane deformations u, v, and w do not affect transverse shear strains and there are no transverse shear strains when a plate cross-section is assumed to be rigid (i.e. the Kirchhoff assumption), transverse shear deformations are due only to cross-section warpings. Consequently, distributions of transverse shear strains can be approximated before solving the whole structural problem in order to obtain the reference plane deformations.

To include cross-section warpings, we assume the displacement field of the ith layer as

$$u_1^{(i)} = u - w_{,x} z + \gamma_5 z + \alpha_1^{(i)} z^2 + \beta_1^{(i)} z^3$$
$$u_2^{(i)} = v - w_{,y} z + \gamma_4 z + \alpha_2^{(i)} z^2 + \beta_2^{(i)} z^3$$
$$u_3^{(i)} = w. \tag{5}$$

Here, $u(x, y, t)$, $v(x, y, t)$, and $w(x, y, t)$ are the displacements of the reference point, t denotes time, $\gamma_4(x, y, t)$ and $\gamma_5(x, y, t)$ are the transverse shear rotation angles at the reference plane, and $\alpha_j^{(i)}(x, y, t)$ and $\beta_j^{(i)}(x, y, t)$ are functions to be determined by using the continuity conditions of in-plane displacements and interlaminar shear stresses and the stress conditions on the bonding surfaces.

It follows from eqns (2) and (5) that the transverse shear strains are

$$\varepsilon_{13}^{(i)} = \gamma_5 + 2\alpha_1^{(i)} z + 3\beta_1^{(i)} z^2 \tag{6}$$
$$\varepsilon_{23}^{(i)} = \gamma_4 + 2\alpha_2^{(i)} z + 3\beta_2^{(i)} z^2.$$

We also assume that there is no delamination. Hence the in-plane displacements u_1 and u_2 and interlaminar shear stresses σ_{13} and σ_{23} are continuous across the interface of any two adjacent laminae. Moreover, it is assumed that there are no applied shearing loads on the bonding surfaces and hence $\sigma_{13} = \sigma_{23} = \varepsilon_{13} = \varepsilon_{23} = 0$ at $z = z_1$ and $z = z_{N+1}$ planes, where N is the total number of layers. Hence, we have

$$\varepsilon_{23}^{(1)}(x, y, z_1, t) = 0$$
$$\varepsilon_{13}^{(1)}(x, y, z_1, t) = 0$$
$$u_1^{(i)}(x, y, z_{i+1}, t) - u_1^{(i+1)}(x, y, z_{i+1}, t) = 0 \quad \text{for} \quad i = 1, \ldots, N-1$$
$$u_2^{(i)}(x, y, z_{i+1}, t) - u_2^{(i+1)}(x, y, z_{i+1}, t) = 0 \quad \text{for} \quad i = 1, \ldots, N-1$$
$$\sigma_{23}^{(i)}(x, y, z_{i+1}, t) - \sigma_{23}^{(i+1)}(x, y, z_{i+1}, t = 0 \quad \text{for} \quad i = 1, \ldots, N-1$$
$$\sigma_{13}^{(i)}(x, y, z_{i+1}, t) - \sigma_{13}^{(i+1)}(x, y, z_{i+1}, t) = 0 \quad \text{for} \quad i = 1, \ldots, N-1$$
$$\varepsilon_{23}^{(N)}(x, y, z_{N+1}, t) = 0$$
$$\varepsilon_{13}^{(N)}(x, y, z_{N+1}, t) = 0. \tag{7}$$

These $4N$ algebraic equations can be used to determine the $4N$ unknowns (i.e. $\alpha_1^{(i)}, \alpha_2^{(i)}, \beta_1^{(i)}, \beta_2^{(i)}$ for $i = 1, \ldots, N$) in terms of γ_4 and γ_5 as

$$\alpha_1^{(i)} = a_{14}^{(i)}\gamma_4 + a_{15}^{(i)}\gamma_5, \quad \alpha_2^{(i)} = a_{24}^{(i)}\gamma_4 + a_{25}^{(i)}\gamma_5$$
$$\beta_1^{(i)} = b_{14}^{(i)}\gamma_4 + b_{15}^{(i)}\gamma_5, \quad \beta_2^{(i)} = b_{24}^{(i)}\gamma_4 + b_{25}^{(i)}\gamma_5 \tag{8}$$

where $a_{kl}^{(i)}$ and $b_{kl}^{(i)}$ are functions of z_j, $\bar{Q}_{44}^{(j)}$, $\bar{Q}_{45}^{(j)}$, and $\bar{Q}_{55}^{(j)}$. Hence it follows from eqns (5) and (8) that

$$u_1^{(i)} = u - w_x z + \gamma_5 g_{15}^{(i)} + \gamma_4 g_{14}^{(i)}$$
$$u_2^{(i)} = v - w_y z + \gamma_4 g_{24}^{(i)} + \gamma_5 g_{25}^{(i)}$$
$$u_3^{(i)} = w \tag{9}$$

where $g_{14}^{(i)}, g_{15}^{(i)}, g_{24}^{(i)}$, and $g_{25}^{(i)}$ are polynomial functions of z, defined as

$$g_{14}^{(i)} \equiv a_{14}^{(i)}z^2 + b_{14}^{(i)}z^3, \qquad g_{15}^{(i)} \equiv z + a_{15}^{(i)}z^2 + b_{15}^{(i)}z^3$$
$$g_{24}^{(i)} \equiv z + a_{24}^{(i)}z^2 + b_{24}^{(i)}z^3, \quad g_{25}^{(i)} \equiv a_{25}^{(i)}z^2 + b_{25}^{(i)}z^3 \tag{10}$$

$g_{15}^{(i)}$ and $g_{24}^{(i)}$ are the so-called shear warping functions, and we call $g_{14}^{(i)}$ and $g_{25}^{(i)}$ shear coupling functions.

Since each $g_{ij}^{(i)}$ is constrained to pass through the origin, it cannot be used to represent an arbitrary cubic function for the ith layer. Hence, to have complete cubic functions, we add two constants to each $g_{ij}^{(i)}$ and assume that

$$g_{14}^{(i)} \equiv c_{14}^{(i)} + d_{14}^{(i)}z + a_{14}^{(i)}z^2 + b_{14}^{(i)}z^3, \quad g_{15}^{(i)} \equiv c_{15}^{(i)} + d_{15}^{(i)}z + a_{15}^{(i)}z^2 + b_{15}^{(i)}z^3$$
$$g_{24}^{(i)} \equiv c_{24}^{(i)} + d_{24}^{(i)}z + a_{24}^{(i)}z^2 + b_{24}^{(i)}z^3, \quad g_{25}^{(i)} \equiv c_{25}^{(i)} + d_{25}^{(i)}z + a_{25}^{(i)}z^2 + b_{25}^{(i)}z^3. \tag{11}$$

It follows from eqn (9) that the strains are

$$\varepsilon_{13}^{(i)} = \gamma_5\, g_{15z}^{(i)} + \gamma_4\, g_{14z}^{(i)}$$
$$\varepsilon_{23}^{(i)} = \gamma_4\, g_{24z}^{(i)} + \gamma_5\, g_{25z}^{(i)}$$
$$\varepsilon_{12}^{(i)} = u_y + v_x - 2w_{xy}z + \gamma_{5y}\, g_{15}^{(i)} + \gamma_{4y}\, g_{14}^{(i)} + \gamma_{4x}\, g_{24}^{(i)} + \gamma_{5x}\, g_{25}^{(i)}$$
$$\varepsilon_{11}^{(i)} = u_x - w_{xx}z + \gamma_{5x}\, g_{15}^{(i)} + \gamma_{4x}\, g_{14}^{(i)}$$
$$\varepsilon_{22}^{(i)} = v_y - w_{yy}z + \gamma_{4y}\, g_{24}^{(i)} + \gamma_{5y}\, g_{25}^{(i)}$$
$$\varepsilon_{33}^{(i)} = 0 \tag{12}$$

where $g_{jkz}^{(i)} \equiv \partial g_{jk}^{(i)}/\partial z$. Substituting eqns (9), (12), and (3) into eqn (7) and then setting the coefficients of γ_4 and γ_5 to zero yield $8N$ algebraic equations. However, we have $16N$ unknowns—$a_{14}^{(i)}$, $a_{15}^{(i)}$, $a_{24}^{(i)}$, $a_{25}^{(i)}$, $b_{14}^{(i)}$, $b_{15}^{(i)}$, $b_{24}^{(i)}$, $b_{25}^{(i)}$, $c_{14}^{(i)}$, $c_{15}^{(i)}$, $c_{24}^{(i)}$, $c_{25}^{(i)}$, $d_{14}^{(i)}$, $d_{15}^{(i)}$, $d_{24}^{(i)}$, $d_{25}^{(i)}$, for $i = 1, \ldots, N$. Hence, we need $8N$ extra equations in order to solve for these constants.

According to the shear warping functions obtained and shown in eqn (10), if the Jth layer contains the origin of the co-ordinate system, one needs to choose

$$c_{14}^{(J)} = c_{15}^{(J)} = c_{24}^{(J)} = c_{25}^{(J)} = d_{14}^{(J)} = d_{25}^{(J)} = 0, \quad d_{15}^{(J)} = d_{24}^{(J)} = 1 \tag{13}$$

which makes the warping functions pass through the origin and γ_4 and γ_5 represent the shear rotation angles at the reference plane. For other layers, one can again choose to use eqns (13). Then, eqns (13) for $i = 1, \ldots, N$ represent the required $8N$ equations. However, the warping functions in eqn (10) are constrained to pass through the origin, and hence they are inappropriate for laminates consisting of few layers.

For isotropic plates, elasticity solutions show that (Timoshenko and Goodier, 1970)

$$\sigma_{13} = G\gamma_5\left(1-\frac{4z^2}{h^2}\right), \quad \sigma_{23} = G\gamma_4\left(1-\frac{4z^2}{h^2}\right). \tag{14}$$

Hence we propose to use

$$c_{14}^{(i)} = d_{14}^{(i)} = a_{15}^{(i)} = a_{24}^{(i)} = c_{25}^{(i)} = d_{25}^{(i)} = 0, \quad d_{15}^{(i)} = d_{24}^{(i)} = 1 \tag{15}$$

for $i = 1, \ldots, J-1, J+1, \ldots N$. Equations (13) and (15) represent the required extra $8N$ equations. However, the form of σ_{13} in eqn (14) is valid only if the y-axis represents the neutral axis of the cross-section on the y–z plane, and the form of σ_{23} is valid only if the x-axis represents the neutral axis of the cross-section on the x–z plane. Hence, for an anisotropic laminate, the neutral axis of the cross-section on the x–z plane may not be on the midplane, the neutral axis of the cross-section on the y–z plane may not be on the midplane, and these two neutral axes may not be on the same plane. These cause difficulties in and reveal complexity of the analysis of anisotropic laminates. However, for symmetric and skew-symmetric laminates, the neutral axes are always on the midplane.

In using the present method of deriving warping functions, the contacting surface of two adjacent layers cannot be used as the reference plane because g_{24z} and g_{15z} are discontinuous at $z = 0$ and hence γ_4 ($= \gamma_4 g_{24z}^{(J)}|_{z=0}$) and γ_5 ($= \gamma_5 g_{15z}^{(J)}|_{z=0}$) are undefined at the $z = 0$ plane. Moreover, if a surface layer contains the reference plane, to release the stiffness of warping functions it is better to divide the layer into two layers; one contains the reference plane, and the other contains the outer surface.

3. SHEAR CORRECTION FACTORS

It follows from eqns (12) and (3) that

$$\begin{Bmatrix} \sigma_{13}^{(i)} \\ \sigma_{23}^{(i)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^{(i)} & \bar{Q}_{45}^{(i)} \\ \bar{Q}_{45}^{(i)} & \bar{Q}_{44}^{(i)} \end{bmatrix} \begin{Bmatrix} g_{15z}^{(i)}\gamma_5 + g_{14z}^{(i)}\gamma_4 \\ g_{24z}^{(i)}\gamma_4 + g_{25z}^{(i)}\gamma_5 \end{Bmatrix}. \tag{16}$$

We point out here that eqn (16) is exact because exact transverse shear strains can always be put in this form, although $g_{jk}^{(i)}$ may be not polynomial functions as those shown in eqn (10) or eqn (11). Hence all derivations presented in this section are exact. To derive the shear correction factors, we consider the form of eqn (16) and assume that the shear stress resultants Q_1 and Q_2 of an equivalent first-order shear deformation theory have the form

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} k_1 A_{55} & k_3 A_{45} \\ k_3 A_{45} & k_2 A_{44} \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_5 + c_1\bar{\gamma}_4 \\ \bar{\gamma}_4 + c_2\bar{\gamma}_5 \end{Bmatrix} \tag{17a}$$

where

$$\begin{aligned} A_{44} &\equiv \int \bar{Q}_{44}^{(i)} dz = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \bar{Q}_{44}^{(i)} dz = \sum_{i=1}^{N} \bar{Q}_{44}^{(i)}(z_{i+1} - z_i) \\ A_{45} &\equiv \int \bar{Q}_{45}^{(i)} dz \\ A_{55} &\equiv \int \bar{Q}_{55}^{(i)} dz. \end{aligned} \tag{17b}$$

Moreover, k_1, k_2, and k_3 are shear correction factors, $\bar{\gamma}_4$ and $\bar{\gamma}_5$ are energy-averaged representatives of γ_4 and γ_5, respectively, c_1 accounts for the shear coupling effect of γ_4 on Q_1, and c_2 accounts for the shear coupling effect of γ_5 on Q_2. Hence there are seven unknowns (i.e. k_1, k_2, k_3, $\bar{\gamma}_4$, $\bar{\gamma}_5$, c_1, c_2) to be determined by matching the shear stress resultants Q_1 and Q_2 and shear strain energy E_n of the derived layerwise higher-order shear theory with those of the equivalent first-order shear theory.

It follows from eqn (16) that

$$Q_1 = \int \sigma_{13}^{(i)} dz = C_{11}\gamma_5 + C_{12}\gamma_4 \tag{18}$$

$$Q_2 = \int \sigma_{23}^{(i)} dz = C_{21}\gamma_4 + C_{22}\gamma_5 \tag{19}$$

$$2E_n = \int (\sigma_{13}^{(i)}\varepsilon_{13}^{(i)} + \sigma_{23}^{(i)}\varepsilon_{23}^{(i)}) dz = \gamma_5^2 C_{31} + 2\gamma_4\gamma_5 C_{32} + \gamma_4^2 C_{33} \tag{20}$$

where

$$\begin{aligned}
C_{11} &= \int (\bar{Q}_{55}^{(i)} g_{15z}^{(i)} + \bar{Q}_{45}^{(i)} g_{25z}^{(i)}) dz \\
C_{12} &= \int (\bar{Q}_{55}^{(i)} g_{14z}^{(i)} + \bar{Q}_{45}^{(i)} g_{24z}^{(i)}) dz \\
C_{21} &= \int (\bar{Q}_{44}^{(i)} g_{24z}^{(i)} + \bar{Q}_{45}^{(i)} g_{14z}^{(i)}) dz \\
C_{22} &= \int (\bar{Q}_{44}^{(i)} g_{25z}^{(i)} + \bar{Q}_{45}^{(i)} g_{15z}^{(i)}) dz \\
C_{31} &= \int (\bar{Q}_{44}^{(i)} g_{25z}^{(i)2} + \bar{Q}_{55}^{(i)} g_{15z}^{(i)2} + 2\bar{Q}_{45}^{(i)} g_{15z}^{(i)} g_{25z}^{(i)}) dz \\
C_{32} &= \int [(\bar{Q}_{45}^{(i)} g_{15z}^{(i)} + \bar{Q}_{44}^{(i)} g_{25z}^{(i)}) g_{24z} + (\bar{Q}_{45}^{(i)} g_{25z}^{(i)} + \bar{Q}_{55}^{(i)} g_{15z}^{(i)}) g_{14z}] dz \\
C_{33} &= \int (\bar{Q}_{44}^{(i)} g_{24z}^{(i)2} + \bar{Q}_{55}^{(i)} g_{14z}^{(i)2} + 2\bar{Q}_{45}^{(i)} g_{14z}^{(i)} g_{24z}^{(i)}) dz.
\end{aligned} \tag{21}$$

It follows from eqn (17a) that

$$Q_1 = (k_1 A_{55} + c_2 k_3 A_{45})\bar{\gamma}_5 + (k_3 A_{45} + c_1 k_1 A_{55})\bar{\gamma}_4 \tag{22}$$

$$Q_2 = (k_2 A_{44} + c_1 k_3 A_{45})\bar{\gamma}_4 + (k_3 A_{45} + c_2 k_2 A_{44})\bar{\gamma}_5 \tag{23}$$

$$\begin{aligned}
2E_n &= Q_1(\bar{\gamma}_5 + c_1\bar{\gamma}_4) + Q_2(\bar{\gamma}_4 + c_2\bar{\gamma}_5) \\
&= \bar{\gamma}_5^2(k_1 A_{55} + 2k_3 A_{45} c_2 + k_2 A_{44} c_2^2) + 2\bar{\gamma}_4\bar{\gamma}_5[k_1 A_{55} c_1 + k_3 A_{45}(1 + c_1 c_2) + k_2 A_{44} c_2] \\
&\quad + \bar{\gamma}_4^2(k_2 A_{44} + 2k_3 A_{45} c_1 + k_1 A_{55} c_1^2).
\end{aligned} \tag{24}$$

Setting the term that contains $\gamma_4(\gamma_5)$ in eqn (18) equal to the term that contains $\bar{\gamma}_4(\bar{\gamma}_5)$ in eqn (22) yields

$$(k_1 A_{55} + c_2 k_3 A_{45})\bar{\gamma}_5 = C_{11}\gamma_5 \tag{25}$$

$$(k_3 A_{45} + c_1 k_1 A_{55})\bar{\gamma}_4 = C_{12}\gamma_4. \tag{26}$$

Similarly, it follows from eqns (19) and (23) that

$$(k_2 A_{44} + c_1 k_3 A_{45})\bar{\gamma}_4 = C_{21}\gamma_4 \tag{27}$$

$$(k_3 A_{45} + c_2 k_2 A_{44})\bar{\gamma}_5 = C_{22}\gamma_5. \tag{28}$$

We also obtain from eqns (20) and (24) that

$$\bar{\gamma}_5^2(k_1 A_{55} + 2k_3 A_{45} c_2 + k_2 A_{44} c_2^2) = C_{31}\gamma_5^2 \tag{29}$$

$$\bar{\gamma}_4\bar{\gamma}_5[k_1 A_{55} c_1 + k_3 A_{45}(1 + c_1 c_2) + k_2 A_{44} c_2] = C_{32}\gamma_4\gamma_5 \tag{30}$$

$$\bar{\gamma}_4^2(k_2 A_{44} + 2k_3 A_{45} c_1 + k_1 A_{55} c_1^2) = C_{33}\gamma_4^2. \tag{31}$$

It follows from eqns (29), (28), and (25) that

$$\bar{\gamma}_5(C_{11} + c_2 C_{22}) = C_{31}\gamma_5. \tag{32}$$

We obtain from eqns (30), (26), and (27) that

$$\bar{\gamma}_5(C_{12} + c_2 C_{21}) = C_{32}\gamma_5. \tag{33}$$

It follows from eqns (30), (28), and (25) that

$$\bar{\gamma}_4(C_{22} + c_1 C_{11}) = C_{32}\gamma_4. \tag{34}$$

From eqns (31), (26), and (27), we obtain

$$\bar{\gamma}_4(C_{21} + c_1 C_{12}) = C_{33}\gamma_4. \tag{35}$$

From eqns (34), (35), (32), and (33), we obtain that

$$c_1 = \frac{C_{21}C_{32} - C_{33}C_{22}}{C_{11}C_{33} - C_{12}C_{32}} \tag{36}$$

$$c_2 = \frac{C_{32}C_{11} - C_{31}C_{12}}{C_{21}C_{31} - C_{22}C_{32}}. \tag{37}$$

It follows from eqns (32) and (35) that

$$\frac{\gamma_5}{\bar{\gamma}_5} = \frac{C_{11} + c_2 C_{22}}{C_{31}} \tag{38}$$

$$\frac{\gamma_4}{\bar{\gamma}_4} = \frac{C_{21} + c_1 C_{12}}{C_{33}}. \tag{39}$$

It also follows from eqns (25), (26), (38), and (39) that the shear correction factors are

$$k_1 = \frac{C_{11}C_{33}(C_{11}+c_2C_{22})-c_2C_{12}C_{31}(C_{21}+c_1C_{12})}{A_{55}C_{31}C_{33}(1-c_1c_2)} \tag{40}$$

$$k_3 = \frac{C_{12}C_{31}(C_{21}+c_1C_{12})-c_1C_{11}C_{33}(C_{11}+c_2C_{22})}{A_{45}C_{31}C_{33}(1-c_1c_2)}. \tag{41}$$

One can also obtain from eqns (27), (28), (38), and (39) that

$$k_2 = \frac{C_{21}C_{31}(C_{21}+c_1C_{12})-c_1C_{22}C_{33}(C_{11}+c_2C_{22})}{A_{44}C_{31}C_{33}(1-c_1c_2)} \tag{42}$$

$$k_3 = \frac{C_{22}C_{33}(C_{11}+c_2C_{22})-c_2C_{21}C_{31}(C_{21}+c_1C_{12})}{A_{45}C_{31}C_{33}(1-c_1c_2)}. \tag{43}$$

It can be proved that the k_3 in eqn (41) is equal to that in eqn (43) by using eqns (38), (39), and (32)–(35).

To use the equivalent first-order shear theory in solving structural problems, one needs to define coupled energy-averaged shear rotation angles $\hat{\gamma}_5$ and $\hat{\gamma}_4$ as

$$\hat{\gamma}_5 \equiv \bar{\gamma}_5 + c_1\bar{\gamma}_4, \quad \hat{\gamma}_4 \equiv \bar{\gamma}_4 + c_2\bar{\gamma}_5. \tag{44}$$

The equivalent displacement field is then

$$\begin{aligned} u_1^{(i)} &= u - w_{,x}z + \hat{\gamma}_5 z \\ u_2^{(i)} &= v - w_{,y}z + \hat{\gamma}_4 z \\ u_3^{(i)} &= w. \end{aligned} \tag{45}$$

Using eqn (45) to derive the first-order shear-deformable plate theory and then solving the governing equations with specified boundary and loading conditions, one can obtain the solutions of u, v, w, $\hat{\gamma}_4$, and $\hat{\gamma}_5$. After the values of $\hat{\gamma}_4$ and $\hat{\gamma}_5$ are obtained, one can obtain from eqn (44) that

$$\bar{\gamma}_5 = \frac{\hat{\gamma}_5 - c_1\hat{\gamma}_4}{1-c_1c_2}, \quad \bar{\gamma}_4 = \frac{\hat{\gamma}_4 - c_2\hat{\gamma}_5}{1-c_1c_2}. \tag{46}$$

One can then use eqns (38) and (39) to obtain γ_4 and γ_5, and one can next use eqns (12) and (16) to obtain transverse shear strains and stresses.

We note that the present method of calculating shear correction factors is not limited by the form of warping functions shown in eqn (11). If exact warping functions are available, this method can be used to obtain exact shear correction factors.

We note that eqns (40)–(43) show that

$$k_3 \neq \sqrt{k_1 k_2} \tag{47}$$

but $k_3 = \sqrt{k_1 k_2}$ is used by some researchers in the literature.

The influence of transverse shear deformations on the in-plane strains [see eqn (12)] is not included in the matching of strain energies. However, if Q_1 and Q_2 are constant, then $\gamma_{4x} = \gamma_{4y} = \gamma_{5x} = \gamma_{5y} = 0$ and the transverse shear strain energy is decoupled from the flexural strain energy. We also note that kinetic energy is not considered in the matching. Since kinetic energy is a function of u, v, and w as well as γ_4 and γ_5, to match the kinetic energies, system responses need to be obtained before the kinetic energies can be determined, which is not practical and the results are problem-dependent. However, since the kinetic energy due to shear warpings is relatively small, using the shear correcting factors obtained by matching the shear strain energy only should result in no significant loss of accuracy.

Free-edge effects can affect the shear warping functions around the edges of a plate if the load distributions on the edges are not the same as those of St. Venant's solutions (Iesan, 1987). However, free-edge effects are neglected here.

4. NUMERICAL RESULTS AND DISCUSSIONS

4.1 *Isotropic and single-layer orthotropic plates*

For an isotropic plate with the midplane being the reference plane, we obtain

$$\begin{aligned}
&\bar{Q}_{44} = \bar{Q}_{55} = G, \bar{Q}_{45} = 0 \\
&g_{15} = g_{24} = z - \frac{4z^3}{3h^2}, \quad g_{14} = g_{25} = 0 \\
&A_{44} = A_{55} = Gh, \quad A_{45} = 0 \\
&C_{11} = C_{21} = G\int_{-h/2}^{h/2} g_{15z}\,dz = \tfrac{2}{3}Gh, \quad C_{12} = C_{22} = C_{32} = 0 \\
&C_{31} = C_{33} = G\int_{-h/2}^{h/2} g_{15z}^2\,dz = G\int_{-h/2}^{h/2} g_{24z}^2\,dz = \tfrac{8}{15}Gh \\
&c_1 = c_2 = 0.
\end{aligned} \tag{48}$$

Substituting eqn (48) into eqns (38)–(40) and (42) yields

$$\frac{\gamma_5}{\bar{\gamma}_5} = \frac{\gamma_4}{\bar{\gamma}_4} = \tfrac{5}{4} \tag{49}$$

$$k_1 = k_2 = \tfrac{5}{6}. \tag{50}$$

For this case, k_3 is indeterminate according to eqns (41) and (43), but $A_{45} = 0$ and hence $A_{45}k_3 = 0$, which is the result needed in using eqn (17a). Equation (50) shows that the shear correction factors obtained are the same as those obtained from the elasticity solution (Reissner, 1945). Pai *et al.* (1993) showed that warping functions of one-layer orthotropic laminates are the same as those in eqn (48).

We point out that γ_4 and γ_5 are geometric shear rotation angles at the midplane and $\bar{\gamma}_4$ and $\bar{\gamma}_5$ represent energy-averaged values of γ_4 and γ_5, respectively. Moreover, $\bar{\gamma}_4$ and $\bar{\gamma}_5$ are different from geometry-averaged shear rotation angles $\breve{\gamma}_4$ and $\breve{\gamma}_5$, which are defined and obtained as

$$\breve{\gamma}_4 \equiv \frac{\int_{-h/2}^{h/2} \gamma_4 g_{24z}\,dz}{h} = \tfrac{2}{3}\gamma_4, \quad \breve{\gamma}_5 \equiv \frac{\int_{-h/2}^{h/2} \gamma_5 g_{15z}\,dz}{h} = \tfrac{2}{3}\gamma_5. \tag{51}$$

The shear correction factors defined in this paper are not exactly the same as those defined by matching only the shear strain energies of the equivalent first-order shear theory and the three-dimensional elasticity theory. If the exact distributions of shear strains of an orthotropic laminate are given, one can use the present method to extract the geometric shear rotation angles γ_4 and γ_5 on the reference plane and the derivatives of shear warping functions as

$$\gamma_4 = \varepsilon_{23}^{(J)}|_{z=0}, \quad \gamma_5 = \varepsilon_{13}^{(J)}|_{z=0}$$

$$g_{14z} = g_{25z} = 0, \quad g_{15z}^{(i)} = \varepsilon_{13}^{(i)}/\gamma_5, \quad g_{24z}^{(i)} = \varepsilon_{23}^{(i)}/\gamma_4. \tag{52}$$

Using eqn (52), one can calculate the coefficients in eqns (21), (36), and (37). One can then calculate the energy-averaged shear rotation angles $\hat{\gamma}_5$ and $\hat{\gamma}_4$ by using eqns (44), (38), and (39) and obtain shear correction factors by using eqns (40)–(43). In using the equivalent first-order shear deformation theory to solve for system responses, boundary conditions for $\hat{\gamma}_4$ and $\hat{\gamma}_5$ can be obtained from the boundary values of γ_4 and γ_5 by using eqns (38), (39), and (44). On the other hand, if only shear strain energies are matched, one does not know how to obtain the values of shear representatives (i.e. $\hat{\gamma}_4$ and $\hat{\gamma}_5$ in the present theory) because they are not defined. It is a common practice in the literature to use either the geometry-averaged shear rotation angles (i.e. $\bar{\gamma}_4$ and $\bar{\gamma}_5$) or the shear rotation angles at the reference plane (i.e. γ_4 and γ_5 in the present theory) as the shear representatives.

If the shear representatives of the isotropic plate considered are chosen to be the geometry-averaged shear rotation angles, that is, $\hat{\gamma}_4 = 2\gamma_4/3$ and $\hat{\gamma}_5 = 2\gamma_5/3$, then matching the exact shear strain energy with that of the equivalent first-order shear theory results in $k_1 = k_2 = \frac{6}{5}$. On the other hand, matching the exact shear resultants Q_1 and Q_2 with those of the equivalent first-order shear theory results in $k_1 = k_2 = 1$. Hence inconsistency occurs. Moreover, because shear representatives are not well defined, one needs to make a subjective choice for their boundary values when the equivalent first-order shear theory is used to solve for system responses. Of course, different choices of boundary values result in different system responses. Hence the present shear correction factors are not exactly the same as those in the literature. Moreover, for practical plate problems, the effect due to incorrect choice of boundary values for the shear representatives is always mixed with the free-edge effect, and hence it is difficult to distinguish them.

Since there are no shear couplings, the shear warping functions represent the cross-section warpings when $\gamma_4 = 1$ and $\gamma_5 = 1$. Assuming $G = 5\times10^5$psi(lbf/in.^{2}) and $h = 0.005$ in., we obtain Fig. 2. Figures 2(a) and 2(b) show that, if the reference plane is shifted from the midplane to a plane at a distance a above the midplane, the shear warping function is just shifted by a constant, the distribution of shear stress is still parabolic, and its maximum still occurs at the midplane. In other words, the shifted warping function $\hat{g}_{15}^{(i)}$ can be obtained from the midplane warping function $g_{15}^{(i)}$ [see eqn (48)] as

$$\hat{g}_{15}^{(i)} = \frac{g_{15}^{(i)}|_{z=z-a}}{g_{15z}^{(i)}|_{z=a}}. \tag{53}$$

Moreover, the shear correction factors do not change, and $\bar{\gamma}_4$ does not change because

$$\bar{\gamma}_4 = \frac{\gamma_4^0}{\gamma_4^0/\bar{\gamma}_4} = \frac{\gamma_4^0(1-4z^2/h^2)_{z=a}}{\gamma_4^0(1-4z^2/h^2)_{z=a}/\bar{\gamma}_4} = \frac{\gamma_4^a}{\gamma_4^a/\bar{\gamma}_4} \tag{54}$$

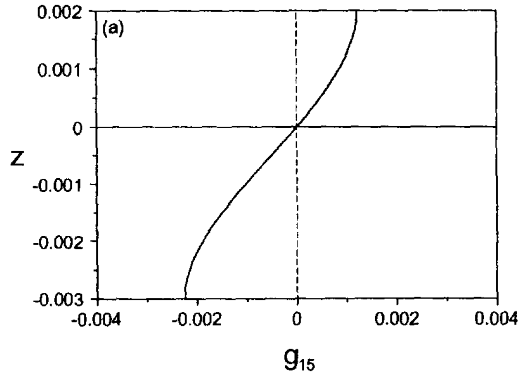
where γ_4^a denotes the shear rotation angle at the $z = a$ plane. Similarly, one can prove that $\bar{\gamma}_5$ does not change when a different reference plane is used. However, the ratios $\gamma_4/\bar{\gamma}_4$ and $\gamma_5/\bar{\gamma}_5$ change because γ_4 and γ_5 represent the shear angles at the current reference plane, not the midplane. Moreover, because the $z = a$ plane is not the neutral plane, extension and shear deformations are coupled even if there are no in-plane loads.

4.2 *Orthotropic laminates*

We consider the orthotropic laminates studied by Pagano (1969, 1970). The properties of these graphite/epoxy layers are

$$E_{11} = 25\times10^6 \text{ psi}, \quad E_{22} = 1\times10^6 \text{ psi}, \quad E_{33} = 1\times10^6 \text{ psi};$$

$$\nu_{12} = 0.25, \quad \nu_{13} = 0.25, \quad \nu_{23} = 0.25;$$



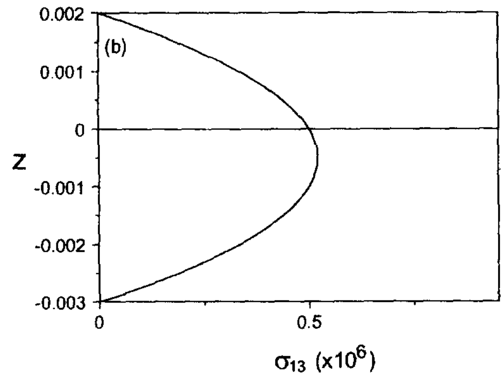


Fig. 2. An isotropic plate: (a) the shear warping function g_{15}, and (b) the transverse shear stress σ_{13}.

$$G_{12} = 0.5 \times 10^6 \text{ psi}, \quad G_{13} = 0.5 \times 10^6 \text{ psi}, \quad G_{23} = 0.2 \times 10^6 \text{ psi};$$

$$\text{thickness}, \hat{t} = 0.005 \text{ in.} \tag{55}$$

Because shear coupling functions g_{14} and g_{25} are obtained to be zero for orthotropic laminates, $\sigma_{13}^{(i)}/\gamma_5 = G_{13}g_{15z}^{(i)}$ and $\sigma_{23}^{(i)}/\gamma_4 = G_{23}g_{24z}^{(i)}$ if $\alpha = 0^\circ$ and $\sigma_{13}^{(i)}/\gamma_5 = G_{23}g_{15z}^{(i)}$ and $\sigma_{23}^{(i)}/\gamma_4 = G_{13}g_{24z}^{(i)}$ if $\alpha = 90^\circ$. Hence the distribution of transverse shear stresses can be obtained without knowing the values of γ_4 and γ_5. For the [0°/90°] laminate, the shear warping function g_{15} and the transverse shear stress σ_{13} (if $\gamma_5 = 1$) are shown in Fig. 3, where the reference plane is chosen to pass through the neutral axis of the cross-section on the y–z plane. In Fig. 3(b), we note that the distribution of σ_{13} obtained (the solid line) is almost the same as the exact one (the broken line) obtained by Pagano [1969, Fig. 3(c)]. Moreover, the shear correction factors obtained ($k_1 = k_2 = 0.8538$) are close to those of Whitney (1973) ($k_1 = k_2 = 0.8212$). We point out here that the comparison shown in Fig. 3(b) (together with others in the following) is qualitative and not quantitative because the solid line in Fig. 3(b) represents σ_{13}/γ_5 but the broken line represents σ_{13} (the horizontal scale is not for this measure) with the specified boundary and loading conditions studied by Pagano (1969).

For the [0°/90°/0°] laminate, the shear warping functions and transverse shear stresses (if $\gamma_4 = \gamma_5 = 1$) are shown in Fig. 4. Since this is a symmetric laminate, the midplane is used as the reference plane. In Fig. 4(b), we note that the distribution of σ_{13} obtained (the solid line) is similar to the one (the broken line) obtained by Pagano [1969, Fig. 4(c)]. In Fig. 4(d), the broken line is obtained by Pagano (1970, Fig. 7). The shear correction factors are obtained as $k_1 = 0.7031$ and $k_2 = 0.8676$; there are no valid data in the literature for comparison.

Next, we consider the orthotropic laminates studied by Noor and Burton (1989b). The material properties are

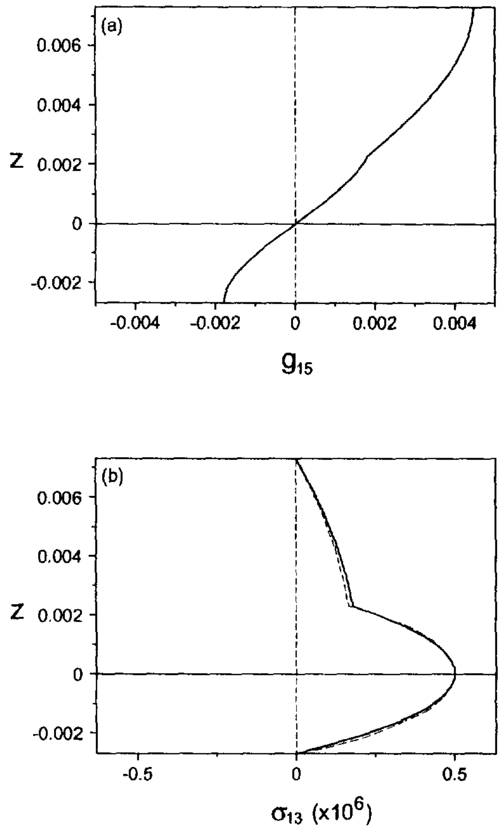


Fig. 3. A [0°/90°] laminate: (a) the shear warping function g_{15}, and (b) the transverse shear stress σ_{13}.

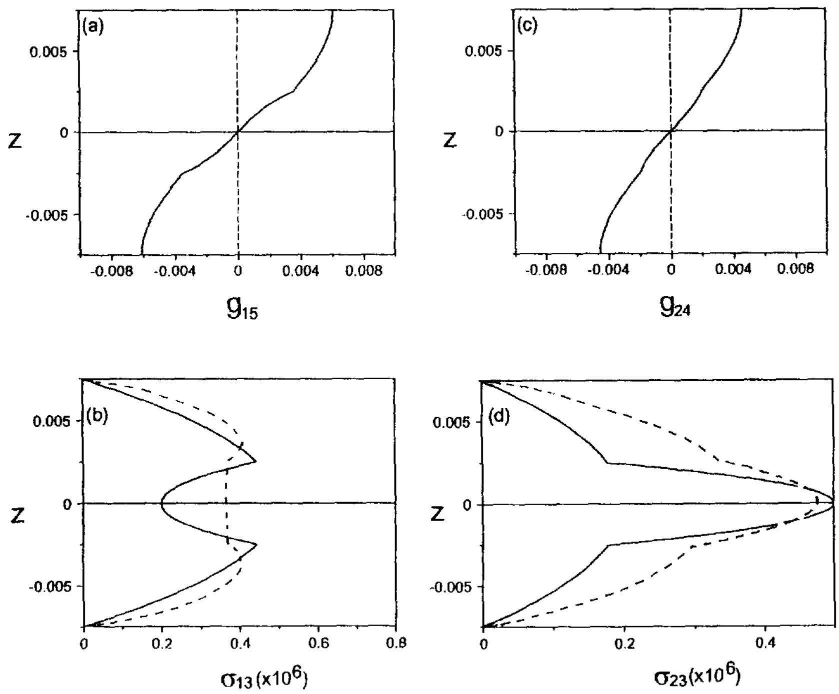


Fig. 4. A [0°/90°/0°] laminate: (a) the shear warping function g_{15}, (b) the transverse shear stress σ_{13}, (c) the shear warping function g_{24}, and (d) the transverse shear stress σ_{23}.

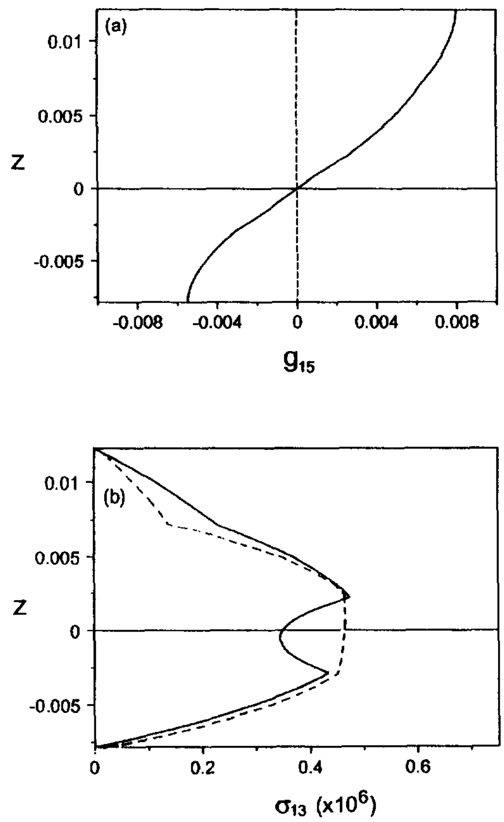


Fig. 5. A [0°/90°/0°/90°] laminate: (a) the shear warping function g_{15}, and (b) the transverse shear stress σ_{13}.

$$E_{11} = 15\times10^6\,\text{psi},\quad E_{22} = 1\times10^6\,\text{psi},\quad E_{33} = 1\times10^6\,\text{psi};$$
$$\nu_{12} = 0.3,\quad \nu_{13} = 0.3,\quad \nu_{23} = 0.49;$$
$$G_{12} = 0.5\times10^6\,\text{psi},\quad G_{13} = 0.5\times10^6\,\text{psi},\quad G_{23} = 0.35\times10^6\,\text{psi};$$
$$\text{thickness}, \hat{t} = 0.005\,\text{in}. \tag{56}$$

For the [0°/90°/0°/90°] laminate, the shear warping function g_{15} and transverse shear stress σ_{13} (if $\gamma_5 = 1$) are shown in Fig. 5. Figure 5(b) demonstrates that the distribution of σ_{13} obtained (the solid line) is similar to the one (the broken line) obtained by Noor and Burton (1989b, Fig. 3). However, the shear correction factors obtained ($k_1 = k_2 = 0.8138$) are different from those of Noor and Burton (1989b) ($k_1 = k_2 = 0.6889$). Since only shear strain energy is matched in defining shear correction factors in the work of Noor and Burton (1989b) but both shear strain energy and shear stress resultants are matched in the present work, the discrepancy in the estimated shear correction factors may be due to the difference in definitions of shear correction factors. Moreover, since the warping functions and shear stress distributions obtained by Pagano (1969) and Noor and Burton (1989b) are dependent on the boundary and loading conditions and the thickness-to-span ratio [see, for example, Figs 4(c) and 4(e) of Pagano (1969)], the discrepancy is also problem-dependent. However, the discrepancy may be because the assumed polynomial warping functions (see eqn (11)) do not fit the actual warping geometry very well. This will be the subject of future work in this area.

For the [0°/90°/0°/90°/0°/90°/0°/90°/0°] laminate, the shear warping functions and transverse shear stresses are shown in Fig. 6. Figure 6(d) shows that the distribution of σ_{23} obtained (the solid line) is similar to the one (the broken line) obtained by Noor and Burton (1989b, Fig. 4). The shear correction factors obtained ($k_1 = 0.8133$, $k_2 = 0.8156$) are close to $k_1 = k_2 = 5/6$, which are expected because an orthotropic laminate should behave like

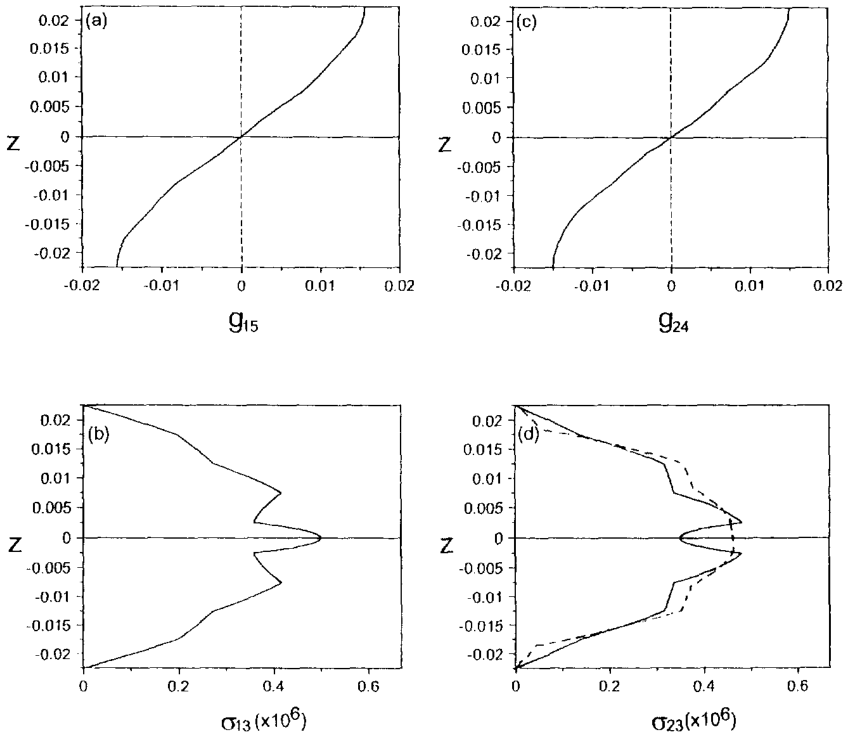


Fig. 6. A [0°/90°/0°/90°/0°/90°/0°/90°/0°] laminate: (a) the shear warping function g_{15}, (b) the transverse shear stress σ_{13}, (c) the shear warping function g_{24}, and (d) the transverse shear stress σ_{23}.

an isotropic one when the number of plies becomes large. Again, the values obtained are different from those of Noor and Burton (1989b) ($k_1 = 0.8365$, $k_2 = 0.7345$).

For a nineteen-layered cross-ply laminate (i.e. [(0°/90°)$_9$/0°]), the shear warping functions and transverse shear stresses are shown in Fig. 7. We note that, although the shear warping functions and shear stresses are zigzag, their global distributions are very similar to those of an isotropic plate (i.e. a parabolic function, see Fig. 2). Moreover, the shear correction factors obtained ($k_1 = 0.8112$, $k_2 = 0.8116$) show that $k_1 \approx k_2$, which is because the number of plies is large and hence the laminate behaves like an isotropic one. The values obtained are close to those of Noor and Burton (1989b) ($k_1 = k_2 = 0.8018$ for a twenty-layered cross-ply laminate).

4.3 *Anisotropic laminates*

We consider angle-ply laminates consisting of layers having material properties shown in eqn (56). For a general angle-ply laminate, because of extension–extension couplings, the neutral axes cannot be located. However, for symmetric and skew-symmetric laminates, the midplane is the neutral plane. For a symmetric [10°/5°/0°/5°/10°] laminate, the shear warping and coupling functions are shown in Fig. 8. We note that shear coupling functions $g_{14}^{(i)}$ and $g_{25}^{(i)}$ are non-trivial and are odd functions. Since shear couplings exist and shear rotation angles γ_4 and γ_5 are unknown before the system responses are obtained, the distributions of shear stresses are unknown. However, the shear correction factors are obtained as $k_1 = 0.8313$, $k_2 = 0.8298$, and $k_3 = 1.3012$. We note that $k_3 \neq \sqrt{k_1 k_2}$, but some researchers used $k_3 = \sqrt{k_1 k_2}$ in analyzing composite laminates. Moreover, $c_1 = 0.0501$ and $c_2 = 0.07462$ [see eqn (17a)], which show the influence of shear couplings on the shear stress resultants.

For a skew-symmetric [10°/5°/0°/−5°/−10°] laminate, the shear warping and coupling functions are shown in Fig. 9. We note that shear coupling functions $g_{14}^{(i)}$ and $g_{25}^{(i)}$ are even functions. The shear correction factors are obtained as $k_1 = 0.6056$, $k_2 = 0.6389$, and $k_3 = 0$ (because $A_{45} = 0$), and $c_1 = c_2 = 0$. If $g_{14}^{(i)}$ and $g_{25}^{(i)}$ were neglected, k_1 would be 0.8337 and

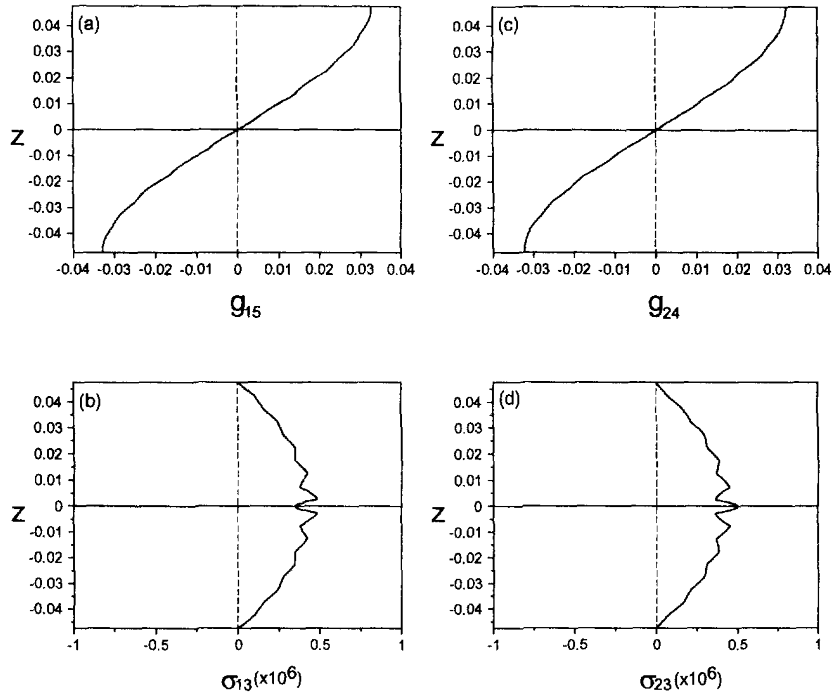


Fig. 7. A nineteen-layered cross-ply laminate ([(0°/90°)$_9$/0°]): (a) the shear warping function g_{15}, (b) the transverse shear stress σ_{13}, (c) the shear warping function g_{24}, and (d) the transverse shear stress σ_{23}.

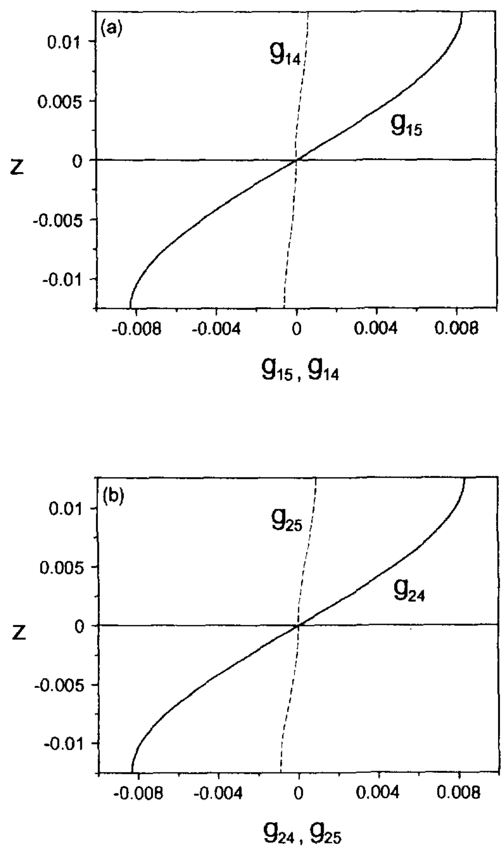


Fig. 8. A [10°/5°/0°/5°/10°] laminate: (a) the shear warping function g_{15} and the shear coupling function g_{14}, and (b) the shear warping function g_{24} and the shear coupling function g_{25}.

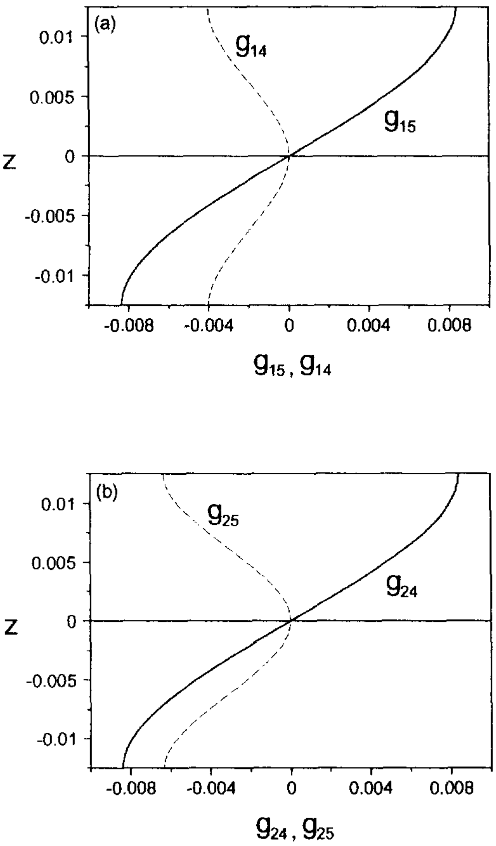


Fig. 9. A [10°/5°/0°/−5°/−10°] laminate: (a) the shear warping function g_{15} and the shear coupling function g_{14}, and (b) the shear warping function g_{24} and the shear coupling function g_{25}.

k_2 0.8321. It shows that shear couplings do affect the values of k_1 and k_2 although $k_3 = c_1 = c_2 = 0$.

For an asymmetric [60°/30°/0°/90°/45°] laminate, the shear warping and coupling functions are shown in Fig. 10. We note that shear coupling functions $g_{14}^{(i)}$ and $g_{25}^{(i)}$ are not even or odd functions. The shear correction factors are $k_1 = 0.6784$, $k_2 = 0.7195$, and $k_3 = 2.1981$, and $c_1 = -0.04086$ and $c_2 = 0.3361$. We note that k_3 is much greater than unity. This shows that it is difficult to estimate the value of k_3 based on our understanding of isotropic and orthotropic plates.

Although the present method of deriving warping functions is not exact, the derived warping functions satisfy the continuity conditions of in-plane displacements and interlaminar shear stresses and the free shear stress conditions on the bonding surfaces. Moreover, the definition of shear correction factors presented in Section 3 is more rigorous than others in the literature. Hence a combination of the new method of deriving warping functions and the new definition of shear correction factors can be used to obtain fairly accurate shear correction factors. Since the range of validity of the first-order shear theory is strongly dependent on the shear correction factors used, the present method can extend the validity of the first-order shear theory in analyzing anisotropic plates. After the system responses are obtained by using the equivalent first-order shear deformation theory, if more accuracy is required, one can use the post-processing technique to solve the three-dimensional elasticity equations or even use the predictor–corrector procedures to refine the solution.

5. CLOSURE

Shear warping functions of a laminate are shown to be dependent on detailed laminate construction and material properties of constituent layers, and they can be approximated

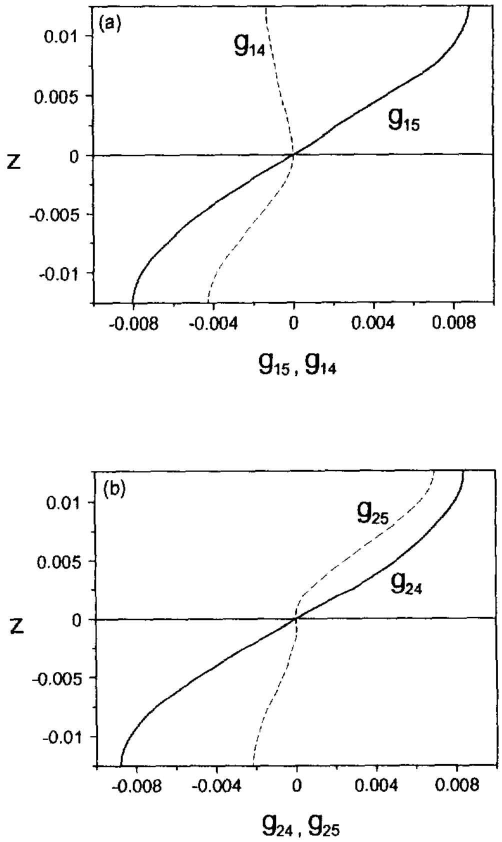


Fig. 10. A [60°/30°/0°/90°/45°] laminate: (a) the shear warping function g_{15} and the shear coupling function g_{14}, and (b) the shear warping function g_{24} and the shear coupling function g_{25}.

by using the continuity conditions of in-plane displacements and interlaminar shear stresses and the free shear stress conditions on the bonding surfaces. The resulting shear deformation theory reveals the elastic coupling effect between two transverse shear deformations of an anisotropic laminate. Moreover, a new method of calculating shear correction factors is derived, in which the shear stress resultants and shear strain energy are conserved. The shear warping functions and shear correction factors obtained are fairly accurate compared with available exact solutions. The present method extends the validity of the first-order shear deformation theory in analyzing anisotropic laminates.

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REFERENCES

Bert, C. W. (1983). Simplified analysis of static shear correction factors for beams of nonhomogeneous cross section. *J. Compos. Mater.* **7**, 525–529.

Bhimaraddi, A. (1984). A higher order theory for free vibration analysis of circular cylindrical shells. *Int. J. Solids Structures* **20**, 623–630.

Chow, T. S. (1971). On the propagation of flexural waves in an orthotropic laminated plate and its response to an impulsive load. *J. Compos. Mater.* **5**, 306–319.

Dennis, S. T. and Palazotto, A. N. (1989). Transverse shear deformation in orthotropic cylindrical pressure vessels using a higher order shear theory. *AIAA J.* **27**, 1441–1447.

Di Sciuva, M. (1987). An improved shear-deformation theory for moderately thick multilayered anisotropic shells and plates. *J. Appl. Mech.* **54**, 589–596.

Dong, S. B. and Tso, F. K. W. (1972). On a laminated orthotropic shell theory including transverse shear deformation. *J. Appl. Mech.* **39**, 1091–1097.

Gulati, J. T. and Essenburg, F. (1967). Effects of anisotropy in axisymmetric cylindrical shells. *J. Appl. Mech.* **34**, 659–666.

Iesan, D. (1987). In *St. Venant's problem. Lecture Notes in Mathematics, No.* 1279 (Edited by A. Dold and B. Eckmann). Springer-Verlag, Berlin, Germany.

Mirsky, I. and Herrmann, G. (1957). Nonaxially symmetric motions of cylindrical shells. *J. Acoust. Soc. Am.* **29,** 1116–1124.

Noor, A. K. and Burton, W. S. (1989a). Assessment of shear deformation theories for multilayered composite plates. *Appl. Mech. Rev.* **42,** 1–13.

Noor, A. K. and Burton, W. S. (1989b). Stress and free vibration analyses of multilayered composite plates. *J. Compos. Struct.* **11,** 183–204.

Noor, A. K. and Burton, W. S. (1990a). Three-dimensional solutions for antisymmetrically laminated anisotropic plates. *J. Appl. Mech.* **57,** 182–188.

Noor, A. K. and Burton, W. S. (1990b). Assessment of computational models for multilayered anisotropic plates. *Comp. Struct.* **14,** 233–265.

Noor, A. K., Burton, W. S. and Peters, J. M. (1990). Predictor–corrector procedure for stress and free vibration analyses of multilayered composite plates and shells. *Comput. Meth. Appl. Mech. Engng* **82,** 341–364.

Pagano, N. J. (1969). Exact solutions for composite laminates in cylindrical bending. *J. Compos. Mater.* **3,** 398–411.

Pagano, N. J. (1970). Exact solutions for rectangular bidirectional composites and sandwich plates. *J. Compos. Mater.* **4,** 20–34.

Pai, P. F., Nayfeh, A. H., Oh, K. and Mook, D. T. (1993). A refined nonlinear model of composite plates with integrated piezoelectric actuators and sensors. *Int. J. Solids Structures* **30,** 1603–1630.

Pai, P. F. and Nayfeh, A. H. (1994). A unified nonlinear formulation for plate and shell theories. *Nonlinear Dynamics*, **6,** 459–500.

Reddy, J. N. and Liu, C. F. (1985). A higher-order shear deformation theory of laminated elastic shells. *Int. J. Engng Sci.* **23,** 319–330.

Reissner, E. (1945). The effect of transverse shear deformation on the bending elastic plates. *J. Appl. Mech.* **2,** *Trans. ASME* **67,** A-69–77.

Timoshenko, S. P. and Goodier, J. N. (1970). *Theory of Elasticity*, 3rd edition. McGraw-Hill, New York, U.S.A.

Voyiadjis, G. Z. and Shi, G. (1991). A refined two-dimensional theory for thick cylindrical shells. *Int. J. Solids Structures* **27,** 261–282.

Whitney, J. M. (1973). Shear correction factors for orthotropic laminates under static load. *J. Appl. Mech.* **40,** 302–304.

Whitney, J. M. (1987). *Structural Analysis of Laminated Anisotropic Plates.* Technomic Publishing Company, Lancaster, PA, USA.

Whitney, J. M. and Sun, C. T. (1974). A refined theory for laminated anisotropic cylindrical shells. *J. Appl. Mech.* **41,** 471–476.

Yang, P. C., Norris, C. H. and Stavsky, Y. (1966). Elastic Wave Propagation in Heterogeneous Plates. *Int. J. Solids Structures* **2,** 665–684.

Zukas, J. A. and Vinson, J. R. (1971). Laminated transversal isotropic cylindrical shells. *J. Appl. Mech.* **38,** 400–407.